第 2 次作业

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Homework 2

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1 Question

1. 路金甫书（第三版）第二章习题 1,2, 3 （这里稳定性用傅里叶方法讨论）

(陆金甫 & 关治, 2016, pp. 43-44)

1. 讨论对流方程

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0,\quad a>0$$

的差分格式

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_j^{n+1}-u_{j-1}^{n+1}}{h}=0$$

的截断误差及稳定性.

2. 题 1 中差分格式改为

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^{n+1}-u_j^{n+1}}{h}=0$$

讨论其截断误差及稳定性。

3. 讨论扩散方程

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2},\quad a>0$$

的差分格式

$$\frac{3}{2}\frac{u_j^{n+1}-u_j^n}{\tau}-\frac{1}{2}\frac{u_j^n-u_j^{n-1}}{\tau}=a\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}$$

的精度及稳定性.

2. 讨论显式龙格-库塔2(RK2)稳定域包含虚轴哪些部分。 针对对流方程 u_t+u_x=0，空间用中心差分，时间用显式龙格-库塔2， 固定网格比 \tau/h的方式让步长趋于0. 讨论该方法稳定性。

2 Solutions

MATH6008 第二次作业 2022.3.14 (due date)

1. 解：截断误差

$$T_j^{n+1} = \frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\tau} + \frac{a}{h}\left(u(x_j,t_{n+1}) - u(x_{j-1},t_{n+1})\right)$$

$$\overset{\text{在}(x_j,t_{n+1})\text{处Taylor展开}}{=\!=\!=} \frac{1}{\tau}\left(-u_t(x_j,t_{n+1})\cdot(-\tau) + O(\tau^2)\right) + \frac{a}{h}\left(-u_x(x_j,t_{n+1})(-h) + O(h^2)\right)$$

$$= (u_t + au_x)\Big|_{(x_j,t_{n+1})} + O(\tau+h)$$

$$= O(\tau+h).$$

用 Fourier 方法讨论稳定性. 格式写成

$$u_j^{n+1} = u_j^n - a\lambda\left(u_j^{n+1} - u_{j-1}^{n+1}\right), \quad \lambda = \tau/h.$$

作 semi-DFT, $(u_j^n \to V_n e^{i\xi jh})$, $\Rightarrow$

$$V_{n+1} = V_n - a\lambda V_{n+1}\left(1 - e^{i\xi h}\right)$$

$$= V_n / \left(1 + a\lambda(1 - e^{i\xi h})\right)$$

$$\Rightarrow G(\xi; h, \tau) = 1/\left(1 + a\lambda(1 - e^{i\xi h})\right)$$

$$\Rightarrow |G| = 1/\left|1 + a\lambda(1-\cos(\xi h)) - ia\lambda\sin(\xi h)\right| \leq 1 \quad (\forall \xi h \in [-\pi,\pi])$$

∴ 不论 λ 取何值，格式均稳定.

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2. 解：截断误差

$$T_j^{n+1} = \frac{1}{\tau}\left(u(x_j,t_{n+1}) - u(x_j,t_n)\right) + \frac{a}{h}\left(u(x_{j+1},t_{n+1}) - u(x_j,t_{n+1})\right)$$

$$= \frac{1}{\tau}\left(-u_t(x_j,t_{n+1})(-\tau) + O(\tau^2)\right) + \frac{a}{h}\left(-u_x(x_j,t_{n+1})(-h) + O(h^2)\right)$$

$$= (u_t + a u_x)\Big|_{(x_j,t_{n+1})} + O(\tau+h)$$

$$= O(\tau+h).$$

格式写成 $u_j^{n+1} = u_j^n - a\lambda(u_{j+1}^{n+1} - u_j^{n+1})$. 作 semi-DFT

$(u_j^n \to V_n e^{i\xi jh}) \Rightarrow V_{n+1} = V_n - a\lambda V_{n+1}(e^{i\xi h} - 1) = V_n / (1 + a\lambda(e^{i\xi h} - 1))$

$\Rightarrow$ 增长因子 $G(\xi; h, \tau) = 1 / [1 + a\lambda(\cos(\xi h) - 1) + i a\lambda \sin(\xi h)]$

$\Rightarrow |G| = [1 + 2a^2\lambda^2 + 2a\lambda((1-a\lambda)\cos(\xi h) - 1)]^{-1/2}$

介于（所遍，当 ξh 取遍 $[-\pi, \pi)$） 1, $|1 - 2a\frac{\tau}{h}|^{-1}$ 之间.

~~要~~ ~~要使~~ ~~$\forall \tau \le \tau_0$~~: ~~$|G| \le 1 + M\tau$~~, ~~至少要有~~ ~~$|G| \le 1$ ($\tau \to 0$)~~

1°. $|1 - 2a\frac{\tau}{h}| \le 1 \iff a\frac{\tau}{h} \in [0,1]$：稳定.

2°. 不能放松上述限制使 $|G| \le 1 + M\tau$ ($\forall \tau \le \tau_0$). Δ?

因 $|1 - 2a\frac{\tau}{h}|^{-1} \le 1 + M\tau$ ($\forall \tau \le \tau_0$) $\Rightarrow$ ~~$2a\frac{\tau}{h} \to$~~

−2−

3. 解：截断误差

$$T_j^{n+1} = \frac{3}{2}\frac{u(x_j,t_{n+1})-u(x_j,t_n)}{\tau} - \frac{1}{2}\frac{u(x_j,t_n)-u(x_j,t_{n-1})}{\tau}$$

$$-\frac{a}{h^2}\left(u(x_{j+1},t_{n+1})-2u(x_j,t_{n+1})+u(x_{j-1},t_{n+1})\right)$$

$$=\frac{1}{\tau}\left(\frac{3}{2}u(x_j,t_{n+1})-2u(x_j,t_n)+\frac{1}{2}u(x_j,t_{n-1})\right)$$

$$-\frac{a}{h^2}\left(u(x_{j+1},t_{n+1})-2u(x_j,t_{n+1})+u(x_{j-1},t_{n+1})\right)$$

$$=\frac{1}{\tau}\Big(-2u_t(x_j,t_{n+1})\cdot(-\tau)+\frac{1}{2}\cdot u_t(x_j,t_{n+1})(-2\tau)$$

$$-2u_{tt}(x_j,t_{n+1})\cdot\left(\frac{\tau^2}{2}\right)+\frac{1}{2}u_{tt}(x_j,t_{n+1})\frac{(2\tau)^2}{2}$$

$$-2u_{t^3}(x_j,t_{n+1})\frac{\tau^3}{6}+\frac{1}{2}\cdot u_{t^3}(x_j,t_{n+1})\cdot\frac{(2\tau)^3}{3}+o(\tau^3)\Big)$$

$$-\frac{a}{h^2}\left(u_{xx}(x_j,t_{n+1})\cdot\frac{h^2}{2}\cdot 2+\frac{1}{4!}u_{x^4}(x_j,t_{n+1})h^4\cdot 2+o(h^4)\right)$$

$$=(u_t-au_{xx})\Big|_{(x_j,t_{n+1})}+O(\tau^2+h^2)$$

∴ 时间精度和空间精度都为 2 阶 ⇒ 具有二阶精度.

引入 $v_j^n = u_j^{n-1}$，$\mu=\tau/h^2$，格式重写为

$$\begin{cases}\left(\frac{3}{2}+2a\mu\right)u_j^{n+1} = 2u_j^n - \frac{1}{2}v_j^n + a\mu\left(u_{j+1}^{n+1}+u_{j-1}^{n+1}\right),\\ v_j^{n+1} = u_j^n.\end{cases}$$

引入 $U_j^n=[u_j^n, v_j^n]^T$，⇒

$$\begin{bmatrix}\frac{3}{2}+2a\mu & 0\\ & 1\end{bmatrix}U_j^{n+1} = \begin{bmatrix}2 & -1/2\\ 1 & \end{bmatrix}U_j^n + \begin{bmatrix}a\mu & \\ & 0\end{bmatrix}\left(U_{j+1}^{n+1}+U_{j-1}^{n+1}\right).$$

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作 semi-DFT ($v_j^n \to v_j^n e^{i\xi jh}$) $\Rightarrow$

$$\begin{bmatrix} \frac{3}{2}+2a\mu & 0 \\ & 1 \end{bmatrix} V_{n+1} = \begin{bmatrix} 2 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} V_n + \begin{bmatrix} a\mu & \\ & 0 \end{bmatrix} V_{n+1} \cdot \left(e^{i\xi h} - e^{-i\xi h}\right)$$

$$\Rightarrow V_{n+1} = \begin{bmatrix} \frac{3}{2}+4a\mu \sin^2\frac{\xi h}{2} & 0 \\ & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1/2 \\ 1 & 0 \end{bmatrix} V_n$$

$\Rightarrow$ 增长矩阵 $$G(\xi;h,\tau) = \begin{bmatrix} \frac{4}{3+8a\mu\sin^2\frac{\xi h}{2}} & \frac{-1}{3+8a\mu\sin^2\frac{\xi h}{2}} \\ 1 & 0 \end{bmatrix}$$

的特征值 p 满足特征方程

$$0 = |pI - G| = \begin{vmatrix} p - \frac{4}{3+8a\mu\sin^2\frac{\xi h}{2}} & \frac{1}{3+8a\mu\sin^2\frac{\xi h}{2}} \\ -1 & p \end{vmatrix}$$

$$= p^2 - \frac{4p}{3+8a\mu\sin^2\frac{\xi h}{2}} + \frac{1}{3+8a\mu\sin^2\frac{\xi h}{2}}$$

$$\Rightarrow p = \frac{\cancel{\sqrt{}}\,2 \pm \sqrt{1-8a\mu\sin^2\frac{\xi h}{2}}}{3+8a\mu\sin^2\frac{\xi h}{2}} \Rightarrow \text{稳定条件？}$$

4. 解. RK2 (显式) 格式:

$$\begin{cases} u_{n+1/2}^{*} = \frac{\tau}{2} f(t_n, u_n) + u_n, \\ u_{n+1} = \tau f(t_{n+1/2}, u_{n+1/2}^{*}) + u_n. \end{cases}$$

将其用于模型问题 $u' = \lambda \cancel{u}\ f(t,u) = \lambda u$, $\Rightarrow \begin{cases} u_{n+1/2}^{*} = \frac{\tau}{2}\cdot \lambda u_n + u_n, \\ u_{n+1} = \tau \cdot \lambda u_{n+1/2} + u_n \end{cases}$

$$\Rightarrow u_{n+1} = \tau\lambda\left(\frac{\tau\lambda}{2}u_n + u_n\right) + u_n = u_n\left[1 + z\left(1+\frac{z}{2}\right)\right]$$

$\Rightarrow$ 稳定域 $$D := \left\{ z \in \mathbb{C} \;\middle|\; \left|1 + z\left(1+\frac{z}{2}\right)\right| \le 1 \right\}$$
$$= \left\{ z = x+iy \;\middle|\; (x^2-y^2+2x+2)^2 + 4y^2(1+x)^2 - 4 \le 0 \right\}.$$

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$$\Rightarrow D \cap \{ z = x + iy \mid x = 0, y \in \mathbb{R} \}_{(\text{虚轴})} = \{ z = iy \mid y^4 \le 0 \}$$
$$= \{ (0,0) \},$$

i.e. 虚轴上仅原点在稳定域中.

$$u_t + u_x = 0 \xRightarrow{\text{空间中心差分}} u_j'(t) + \frac{u_{j+1}(t) - u_{j-1}(t)}{2h} = 0$$

$$\xRightarrow{\text{显式RK2}} \begin{cases} u_j^{*n+\frac{1}{2}} = u_j^n + \frac{\tau}{2}\left[-\frac{u_{j+1}^n - u_{j-1}^n}{2h} \right], \\ u_j^{n+1} = u_j^n + \tau \left[-\frac{u_j^{*n+\frac{1}{2}} + u_j^{*n+\frac{1}{2}}}{2h} \right] \end{cases}$$

$$\Rightarrow u_j^{n+1} = u_j^n - \frac{\tau}{2h} \left\{ \left[u_{j+1}^n - \frac{\tau}{2} \frac{u_{j+2}^n - u_j^n}{2h} \right] - \left[u_{j-1}^n - \frac{\tau}{2} \frac{u_j^n - u_{j-2}^n}{2h} \right] \right\}$$
$$= \frac{\tau^2}{8h^2} u_{j-2}^n + \frac{\tau}{2h} u_{j-1}^n - \frac{\tau^2}{4h^2} u_j^n - \frac{\tau}{2h} u_{j+1}^n + \frac{\tau^2}{8h^2} u_{j+2}^n .$$

$U_j^n = (u_j^n)_{j=0,1,\cdots}$, 格式写成 $U_j^{n+1} = A U_j$, (周期边界) $\Rightarrow$

$$A = \begin{bmatrix} -\frac{\tau}{4h^2} & -\frac{\tau}{2h} & \frac{\tau^2}{8h^2} & 0 & \cdots & & & \cdots & \frac{\tau^2}{8h^2} & \frac{\tau}{2h} \\ \frac{\tau}{2h} & -\frac{\tau^2}{4h^2} & -\frac{\tau}{2h} & \frac{\tau^2}{8h^2} & 0 & \cdots & & & 0 & \frac{\tau^2}{8h^2} \\ \frac{\tau^2}{8h^2} & \frac{\tau^2}{2h} & -\frac{\tau^2}{4h^2} & -\frac{\tau}{2h} & \frac{\tau^2}{8h^2} & 0 & \cdots & & 0 & 0 \\ 0 & \frac{\tau^2}{8h^2} & \cdots & & & & & & & \\ & & & \cdots (\text{循环}) & & & & & & \end{bmatrix}$$

$\Rightarrow$ A 的特征值 $\lambda_p = \cdots$

$\Rightarrow$ $\boxed{\text{稳定性}}$, 必须 $\lambda_p \tau \in D$, $\forall p$, $\forall \tau \le \tau_0$.

$\Rightarrow$ τ 应满足…,

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References

陆金甫, & 关治. (2016). *偏微分方程数值解法* (3 ed.). 清华大学出版社.