第 3 次作业

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摘　要：截止日期：2022-04-04.
关键词：词 1，词 2

Homework 3

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目 录

1 Question

3. 试求出解 Poisson 方程 $-\Delta u = f(x,y)$ 的差分格式

$$-(u_{i+1,j+1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i-1,j-1}-4u_{ij}) = 2h^2 f_{ij}$$

的截断误差.

5. 用柱坐标表示的 Poisson 方程的形式为

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2} = -f(r,\varphi,z),$$

其中 $r=\sqrt{x^2+y^2}$，$\tan\varphi=\frac{y}{x}$. 试写出逼近上述方程的一个差分格式.

4. 在 $D=\{(x,y)\,|\,0\leqslant x,y\leqslant 1\}$ 上给出边值问题

$$\begin{cases} -\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=16, & 0<x,y<1, \\ u\big|_{x=1}=0, \dfrac{\partial u}{\partial y}\Big|_{y=1}=-u, & \\ \dfrac{\partial u}{\partial x}\Big|_{x=0}=\dfrac{\partial u}{\partial y}\Big|_{y=0}=0. & \end{cases}$$

取 $h=\frac{1}{4}$，试用五点差分格式求此问题的数值解.

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其中 $D=\{(x,y)\,|\,|x|,|y|<1\}$.

(1) 用正方形网格($k=h$)列出相应的差分方程；

(2) 对 $h=1, h=\frac{1}{2}$ 分别求解.

2. 列出用五点差分格式求解 Laplace 方程的第三边值问题

$$\begin{cases} \Delta u = 0, & 0<x,y<1, \\ u_x - u = 1+y, & x=0, 0\leqslant y\leqslant 1, \\ u_x + u = 2-y, & x=1, 0\leqslant y\leqslant 1, \\ u_y - u = -1-x, & y=0, 0\leqslant x\leqslant 1, \\ u_y + u = -2+x, & y=1, 0\leqslant x\leqslant 1 \end{cases}$$

($h=k=\frac{1}{2}$)的线性代数方程组.

3. 试求出解 Poisson 方程 $-\Delta u = f(x,y)$ 的差分格式

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$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2} = -f(r,\varphi,z),$$

其中 $r=\sqrt{x^2+y^2}$，$\tan\varphi=\frac{y}{x}$. 试写出逼近上述方程的一个差分格式.

(陆金甫 ＆ 关治, 2016, p. 154)

Numerical methods for PDEs: HW3

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1 Theory

(10 pts)

1. (路金甫，第三版) 第五章习题 3, 5.

2. Show that the discretization

$$-\frac{\kappa_{j+1/2}(u_{j+1}-u_j)-\kappa_{j-1/2}(u_j-u_{j-1})}{h^2}=f(x_j)$$

is of a second order accuracy for the 1D equation $-(\kappa u')'=f(x)$. Show that the resulted matrix is symmetric (assuming Dirichlent boundary condition).

3. Consider the equation in 2D:

$$u-\Delta u=f, \quad u|_{\partial\Omega}=0.$$

Suppose Ω is a square (正方形). Show the maximal principle of the five point scheme:

$$u_{ij}-\Delta_h u_{ij}=f_{ij}.$$

Use this to show the convergence of the method.

4. (Bonus. Not required to submit. Credits obtained can be added to other homework scores provided that the score is no bigger than the total score) Consider the 1D elliptic problem

$$-u''(x)=f(x), \quad u'(0)=u'(1)=0.$$

Assume that $\int_0^1 f(x)\,dx=0$.

- Show that the eigenfunctions of $-u''(x) = \lambda u$, $u'(0) = u'(1) = 0$ are given by $u_n(x) = \cos(n\pi x)$.
- The solution can be written as

$$u = \sum_n c_n u_n(x), \quad f(x) = \sum_n d_n u_n(x).$$

  Given the expression of d_n, explain how to find c_n. This is the cosine transform.

- Conisder discrete case. We set $x_j = jh - \frac{h}{2}$, $j = 1, \cdots, m$ for $h = \frac{1}{m}$. For the FDM

$$-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f_j, \quad \frac{u_1 - u_{-1}}{h} = 0, \quad \frac{u_{m+1} - u_m}{h} = 0,$$

  one can show that $\{u_j\}$ can be written as linear combination of $\cos(n\pi x_j)$. Explain how you can mimick the continuous case to solve this FDM. Explain why it can be fast.

2 Numerics

(10 pts)

1. (路金甫，第三版) 第五章习题4.
2. Consider the mixed type boundary value problem:

$$u'' + u = f, 0 \le x \le \pi$$
$$u'(0) - u(0) = 0, \quad u'(\pi) + u(\pi) = 0$$

   Construct a second order accurate FDM. For $f = -e^x$, plot the error versus the spatial step h in loglog scale.

3. Solve the nonlinear equation $\theta'' = -\sin(\theta)$, $\theta(0) = \alpha, \theta(1) = \beta$. For the nonlinear system of equations you obtained, use Newton's iteration to solve. Plot the error versus h.

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4. Consider the 2D elliptic equation

$$-(a(x,y)u_x)_x - (a(x,y)u_y)_y = f(x,y), \Omega = [-1,1] \times [-1,1]$$
$$u = 0, \ on \ \partial\Omega$$

$a = 1 + 3\exp(-3(x+y)^2 - (x-y)^2)$ and $f = 1$. Apply a five-point scheme for this equation. Determine the order of accuracy of your scheme, using the calculation with a small h as the 'exact' solution.

Hint: To solve the linear system $AU = F$, one option is to construct A directly and do $A\backslash F$. Here, the coefficient is not a constant, constructing this matrix might be a little tricky (remember to keep the matrix sparse in Matlab) (One bad way is to set the point value of U to be 1 at a single point, then output the action of the scheme on this U, which will be the corresponding column of your matrix.) Another better option is to write a function that returns AU when the input is U and then apply an iterative method (such as conjugate gradient) to find the solution. By doing this, you do not have to construct A.

2 Solutions

2.1 Theory

MATH6008 Homework 3 2022.04.04 (due date)

(主教材 p.154) 3. 解：记 $\xi := h\frac{\partial}{\partial x}$, $\eta := h\frac{\partial}{\partial y}$. 在 (x_i, y_j) 处 Taylor 展开：

$$\Rightarrow 2h^2 T_{ij} := -\left(e^{\xi+h} + e^{\xi-h} + e^{-\xi+\eta} + e^{-\xi-\eta} - 4\right) u(x_i, y_j) - 2h^2 f(x_i, y_j)$$

$$= -\left(4 + \frac{2}{2!}(\xi^2 + h^2)\times 2 + O(h^4) - 4\right)u - 2h^2 f$$

$[(\xi^2+\eta^2)u = h^2\Delta u = -h^2 f]$

$$= O(h^4).$$

$$\Rightarrow T_{ij} = O(h^2).$$

(主教材 p.154) 5. 解：[参考 Lec6 讲义 p.9]

记 $(\delta_r u)_i := \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta r}$, $r_i = (i+1/2)\Delta r$, $\theta_j = j\Delta\theta$, $z_k = k\Delta z$.

$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ 离散为 $\frac{1}{r_i}\frac{\partial}{\partial r}(r_i \delta_r u_i) \Rightarrow \frac{1}{r_i}\frac{1}{\Delta r}\left[r_{i+1/2}(\delta_r u)_{i+1/2,j} - r_{i-1/2}(\delta_r u)_{i-1/2,j}\right]$

$$= \frac{1}{r_i}\frac{1}{\Delta^2 r}\left[r_{i+1/2}(u_{i+1,j} - u_{ij}) - r_{i+1/2}(u_{ij} - u_{i-1,j})\right]$$

∴ 原方程离散为：

$$\frac{1}{r_i \Delta^2 r}\left[r_{i+1/2}u_{i+1,j} - (r_{i+1/2} + r_{i-1/2})u_{ij} + r_{i-1/2}u_{i-1,j}\right]$$

$$+ \frac{1}{r_i^2 \Delta^2\theta}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right]$$

$$+ \frac{1}{\Delta^2 z}\left[u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}\right] = -f_{i,j,k}, \quad i>0. \qquad (1)$$

下面用 "ghost point" 法~~施导 B.C.: $u|_{r=0} = 0$.~~

处理 $r=0$ ($i=0$) 处的边界.

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假定 (1) 在 $i=0$ 处仍成立：

$$\frac{2}{\Delta^2 r}\left[\Delta r\cdot(u_{1j}-u_{0j})-r_{-1/2}(u_{0j}-u_{-1,j})\right]+\cdots=f_{0,j,k}. \qquad (2)$$

注意到 $\lim\limits_{r\to 0} r\frac{\partial u}{\partial r}=0$，我们假定

$$r_{0-1/2}\frac{u_{0j}-u_{-1,j}}{\Delta r}=0, \qquad (3)$$

将 (3) 代入 (2) 得

$$\frac{2}{\Delta^2 r}\left(u_{1,j}-u_{0j}\right)+\cdots=0. \qquad (4)$$

∴ 柱坐标表示的 Poisson 方程：$\Delta_3 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2}=-f(r,\varphi,z)$

的一个差分格式为：$\begin{cases}\text{式(1)}, & i>0,\\ \text{式(4)}, & i=0.\end{cases}$

(Hw3) 2. 解：记 $\xi:=h\frac{d}{dx}$，则

$$\begin{aligned}
h^2T_i:&=-e^{\frac{\xi}{2}}k\cdot(e^{\xi}-1)u+e^{-\frac{\xi}{2}}k\cdot(1-e^{-\xi})u-f\\
&=-\left[\left(1+\frac{\xi}{2}+\frac{\xi^2}{8}\right)k+O(h^3)\right]\left[\left(\xi+\frac{\xi^2}{2}+\frac{\xi^3}{6}\right)u+O(h^4)\right]\\
&\quad+\left[\left(1-\frac{\xi}{2}+\frac{\xi^2}{8}\right)k+O(h^3)\right]\left[\left(\xi-\frac{\xi^2}{2}+\frac{\xi^3}{6}\right)u+O(h^4)\right]-h^2f\\
&=-k\cdot\xi^2u-\frac{\xi}{2}k\cdot(2\xi u)+O(h^4)-h^2f\\
&=-h^2(ku''+k'u')+O(h^4)-h^2f=-h^2(ku')'-h^2f+O(h^4)\\
&=O(h^4)\Rightarrow T_i=O(h^2).\Rightarrow \text{具有二阶精度.}
\end{aligned}$$

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(接上) 2. 格式： $AU = F$, $U = (u_1, u_2, \cdots, u_n)^T$.

第 j 行 ($j=1,\cdots,n$)：
$$\begin{cases} -K_{j-1/2}u_{j-1} + (K_{j-1/2}+K_{j+1/2})u_j - K_{j+1/2}u_{j+1} = h^2 f(x_j). \\ u_0 = u_{n+1} = 0, \end{cases}$$

∴ A 的元素：$a_{j,j-1} = -K_{j-1/2}$, ($j=2,\cdots,n$) $a_{jj} = K_{j-1/2}+K_{j+1/2}$, ($j=1,\cdots,n$)

$a_{j,j+1} = -K_{j+1/2}$, ($j=1,\cdots,n-1$), 其余为零.

F 的元素：$f_i = h^2 f(x_i)$, $i=1,\cdots,n$.

∴ A 为对称矩阵 ($a_{ij} = a_{ji}$, $\forall i,j=1,\cdots,n$).

Δ 3. 解：$\Delta_h u_{ij} = u_{ij} - f_{ij}$. 记有 ϕ: $\Delta_h \phi_{ij} = 1$, $\Omega = [0,a]^2$.

则有 $\Delta_h\left(u_{ij} + \phi_{ij}(\|u\|_\infty + \|f\|_\infty)\right) = u_{ij} - f_{ij} + \|u\|_\infty - \|f\|_\infty \geq 0$.

$\Rightarrow$ 极值原理：$u_{ij} + \phi_{ij}(\|u\|_\infty + \|f\|_\infty) \leq \max_{\partial D_h}\{ \overset{0}{u_{ij}} + \phi_{ij}(\|u\|_\infty + \|f\|_\infty)\}$

$u|_{\partial\Omega} = 0$

$\Rightarrow u_{ij} \leq 2\|\phi\|_\infty(\|u\|_\infty + \|f\|_\infty)$.

取 $\phi_{ij} = \frac{1}{4}\left((x_i - \frac{a}{2})^2 + (y_j - \frac{a}{2})^2\right)$, 则 $\Delta_h \phi_{ij} = 1$, $\|\phi\|_\infty = \frac{a^2}{8}$.

$\Rightarrow u_{ij} \leq \left(\frac{a^2}{4}\right)\|u\|_\infty + \frac{a^2}{4}\|f\|_\infty$.

$\Rightarrow (1 - \frac{a^2}{4})\|u\|_\infty \leq \frac{a^2}{4}\|f\|_\infty \Rightarrow \cdots$ 稳定 $\overset{相容}{\Rightarrow}$ 稳定.

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4. 解：(1) 特征函数就是 $\begin{cases} -u'' = \lambda u, \\ u'(0) = u'(1) = 0 \end{cases}$ 的非平凡（非零）解.

1° $\lambda < 0$. 通解 $u = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x)$.

由 B.C.: $u'(0) = u'(1) = 0 \Rightarrow A = B = 0 \Rightarrow u \equiv 0$. 舍去.

2° $\lambda = 0$. 通解 $u = Ax + B \xRightarrow{B.C.} u = B,\ B \neq 0$.

3° $\lambda > 0$. 通解 $u = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. 由 B.C. 得

$B = 0$, ~~λ可开~~ $\sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda_n} = n\pi,\ n = 1, 2, \cdots$.

$\Rightarrow u_n = A_n \cos(n\pi x),\ n = 1, 2, \cdots$.

综上，特征函数为 $u_n = \cos(n\pi x),\ n = 0, 1, \cdots$.

(2) $\{u_n(x),\ x \in [0,1]\} = \{\cos(n\pi x)\}$ 是 $[0,1]$ 上的完备正交函数集:

$$\int_0^1 u_n(x) u_m(x)\,dx = \begin{cases} 0, & n \neq m, \\ 1/2, & n = m. \end{cases}$$

$$f(x) = \sum_n d_n u_n(x) \Rightarrow \int_0^1 f \cdot u_n\,dx = \int_0^1 dx \cdot u_n \sum_m d_m u_m$$

$$= \sum_m d_m \cdot \int_0^1 u_n u_m\,dx = \frac{1}{2} d_n \Rightarrow d_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx.$$

设 $u = \sum_n c_n \cos(n\pi x)$. 代入方程: $-u'' = f$, $\Rightarrow \sum_n c_n (n\pi)^2 \cos(n\pi x)$

$= \sum_n d_n \cos(n\pi x) \xRightarrow{\{u_n\}正交性} c_n = \frac{d_n}{(n\pi)^2},\ n = 1, 2, \cdots$.

c_0 由解规范性条件 $\int_0^1 u\,dx = 0$ 确定: $c_0 = 0$.

(3) 先对 f_j 作快速余弦变换，得 $\hat{f}_n$: $f_j = \sum_n \hat{f}_n \cos(n\pi x_j)$.

$\Rightarrow u_j = \sum_n \frac{\hat{f}_n}{(n\pi)^2} \cos(n\pi x_j)$. 此法的时间复杂度 $O(\quad)$,

快于常规FDM法的 $O(\quad)$.

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2.2 Numerics

2.2.1 Question 2.1

4. 在 $D=\{(x,y)\mid 0\leqslant x,y\leqslant 1\}$ 上给出边值问题

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=16, \quad 0<x,y<1, \\ u\big|_{x=1}=0, \frac{\partial u}{\partial y}\Big|_{y=1}=-u, \\ \frac{\partial u}{\partial x}\Big|_{x=0}=\frac{\partial u}{\partial y}\Big|_{y=0}=0. \end{cases}$$

取 $h=\frac{1}{4}$，试用五点差分格式求此问题的数值解.

2.2.2 Question 2.2

2.2.3 Question 2.3

2.2.4 Question 2.4

References

陆金甫, & 关治. (2016). *偏微分方程数值解法* (3 ed.). 清华大学出版社.