第 3 次作业

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Homework 3

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(p.60) 13. Soln: $\Theta = \begin{cases} 0, & \text{heads}, \\ 1, & \text{tails}. \end{cases}$ $P(\Theta=0)=P(\Theta=1)=\frac{1}{2}$.

$X|\Theta=0 \sim \text{Unif}(0,1)$, $X|\Theta=1 \sim \text{Unif}(3,4)$

$\Rightarrow E(X|\Theta=0)=\frac{1}{2}$, $E(X|\Theta=1)=\frac{7}{2}$, $V(X|\Theta=0)=V(X|\Theta=1)=\frac{1}{12}$.

(1) $EX = E[E(X|\Theta)] = P(\Theta=0)E(X|\Theta=0)+P(\Theta=1)E(X|\Theta=1)$

$= 1/2\,(1/2) + 1/2 \cdot 7/2 = 2.$

(2) $E[V(X|\Theta)] = P(\Theta=0)V(X|\Theta=0) + P(\Theta=1)V(X|\Theta=1)$

$= \frac{1}{2}\cdot\frac{1}{12} + \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{12}$,

$V[E(X|\Theta)] = P(\Theta=0)[E(X|\Theta=0) - E(E(X|\Theta))]^2$

$+ P(\Theta=1)[E(X|\Theta=1) - E(E(X|\Theta))]^2$

$= \frac{1}{2}\cdot[\frac{1}{2}-2]^2 + \frac{1}{2}\cdot[\frac{7}{2}-2]^2 = \frac{9}{4}$.

$\Rightarrow V(X) = E[V(X|\Theta)] + V[E(X|\Theta)] = \frac{1}{12}+\frac{9}{4} = \frac{7}{3}$.

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(p.60) 17. Proof:

$$V(Y) = E(Y-EY)^2 = E\left[(Y-E(Y|X)) + (E(Y|X)-EY)\right]^2$$

$$= \underbrace{E\left[Y-E(Y|X)\right]^2}_{\mathrm{I.}} + \underbrace{E\left[E(Y|X)-EY\right]^2}_{\mathrm{II.}} + \underbrace{2E\left[(Y-E(Y|X))(E(Y|X)-EY)\right]}_{\mathrm{III.}}$$

其中，

$$\mathrm{I.} = E\left\{ E\left[(Y-E(Y|X))^2 \,|\, X \right] \right\} = EV(Y|X).$$

$$\mathrm{II.} = E\left[E(Y|X) - E(E(Y|X)) \right]^2 = VE(Y|X).$$

$$\mathrm{III.} = 2E\left\{ E\left[(Y-E(Y|X))(E(Y|X)-EY) \,|\, X \right] \right\}$$

$$= 2E\left\{ (E(Y|X)-EY) \underbrace{E\left[(Y-E(Y|X)) \,|\, X \right]}_{= E(Y|X)-E(Y|X) = 0} \right\} = 0$$

$$\therefore V(Y) = \mathrm{I} + \mathrm{II} + \mathrm{III} = EV(Y|X) + VE(Y|X).$$

qed.

(p.61) 18. Proof. $E(X|Y=y) = c \ (\forall y) \Rightarrow EX = E[E(X|Y)] = C$,

$$E(XY) = E[E(XY|Y)] = E[YE(X|Y)] = cEY$$

$$\Rightarrow cov(X,Y) = E[(X-EX)(Y-EY)] = E(XY) - EX\cdot EY = cEY - cEY = 0.$$

$\Longleftrightarrow$ X and Y are uncorrelated.

qed.

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(p.61) 21. Proof. $E(Y|X) = X \Rightarrow EY = E[E(Y|X)] = EX$

$\Rightarrow cov(X,Y) = E(XY) - EX \cdot EY = E[E(XY|X)] - EX \cdot EX$

$= E[XE(Y|X)] - (EX)^2 = E[X^2] - (EX)^2 = V(X)$. q.e.d.

(p.61) 22. Soln. (Y,Z)的联合分布列：

$Y \backslash Z$	0	1	
0	0	$1-b$	$1-b$
1	a	$b-a$	b
	a	$1-a$	

(1) $P(Y=0, Z=0) = 0 \neq P(Y=0)P(Z=0)$

$\Rightarrow Y, Z$ 不独立.

(2)

$P(Y=y|Z=0) = \begin{cases} 0, & y=0, \\ 1, & y=1, \end{cases}$ $P(Y=y|Z=1) = \begin{cases} \frac{1-b}{1-a}, & y=0, \\ \frac{b-a}{1-a}, & y=1. \end{cases}$

$\Rightarrow E(Y|Z=0) = 0 \cdot 0 + 1 \cdot 1 = 1,$

$E(Y|Z=1) = 0 \cdot \frac{1-b}{1-a} + 1 \cdot \frac{b-a}{1-a} = \frac{b-a}{1-a}$

$\Rightarrow E(Y|Z) = P(Z=0)E(Y|Z=0) + P(Z=1)E(Y|Z=1)$

$= a \cdot 1 + (1-a) \cdot \frac{b-a}{1-a} = b.$

(p.61) 23. Soln: (i) Poisson.

$X \sim Poisson(\lambda) \Leftrightarrow P(X=x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x \in N$

$\Rightarrow$ MGF: $\psi_X(t) = E(e^{tX}) = \sum_{x=0}^{+\infty} e^{tx} \cdot e^{-\lambda} \frac{\lambda^x}{x!}$

$= e^{-\lambda} \sum_{x=0}^{+\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$.

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(续前页) (ii). Normal. $X \sim N(\mu, \sigma^2)$.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\Rightarrow \psi_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$\overset{x=x-\mu}{=\!=} \frac{1}{\sigma\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{\mathbb{R}} \exp\left[-\frac{(x-\sigma^2 t)^2}{2\sigma^2}\right] d\frac{x-\sigma^2 t}{\sqrt{2}\sigma} \cdot \sqrt{2}\sigma$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}, \quad t \in \mathbb{R}.$$

(iii) Gamma. [~~无法计算~~] $X \sim Gamma(\alpha, \beta)$

(p.30) $$f_X(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

$$\Rightarrow \psi_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx$$

$$= \int_{\mathbb{R}} \frac{x^{\alpha-1} e^{-x/\frac{\beta}{1-\beta t}}}{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha) \cdot (1-\beta t)^\alpha} dx \quad (t < 1/\beta).$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha \cdot \int_{\mathbb{R}} f_{Gamma(\alpha, \frac{\beta}{1-\beta t})}(x)\, dx = \left(\frac{1}{1-\beta t}\right)^\alpha.$$

(p.61) 24. Proof. $X_i \overset{iid}{\sim} Exp(\beta) \Rightarrow f_{X_i}(x) = \frac{1}{\beta} e^{-x/\beta}, \ x > 0$

$$\Rightarrow \psi_{X_i}(t) = E(e^{tX_i}) = \int_0^{+\infty} e^{tx} \frac{1}{\beta} e^{-x/\beta} dx = \frac{1}{1-\beta t}, \quad t < 1/\beta.$$

$X := \sum_i X_i$

$$\Rightarrow \psi_X(t) = E\left(e^{t\sum_i X_i}\right) = E\left(\prod_{i=1}^n e^{tX_i}\right) \overset{iid}{=\!=} \prod_{i=1}^n \psi_{X_i}(t) = \left(\frac{1}{1-\beta t}\right)^n, \quad t < 1/\beta.$$

这是 Gamma(n, β) 的 MGF. $\Rightarrow \sum_{i=1}^n X_i \sim Gamma(n, \beta)$.

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References